Semiseparable matrices and the symmetric eigenvalue problem

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Outline

Introduction

- Eigenvalue problem
- Semiseparable matrices
- Representation

The reduction to semiseparable form

- The reduction algorithms
- The convergence properties

QR-algorithms for semiseparable matrices

- The implicit and the explicit approach
- The implicit QR-algorithms
- Complete algorithm via semiseparable matrices

Future work

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Symmetric eigenvalue problem

Suppose a symmetric matrix A is given. When

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Suppose a symmetric matrix A is given. When

$$Av = \lambda v, \ v \neq 0,$$

λ is an eigenvalue of A corresponding to the eigenvector v. The eigenvalue (spectral) decomposition of the symmetric matrix A:

$$AV = V\Lambda \quad \text{with} \quad \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\text{and} \quad V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \quad (V^T V = I)$$

Examples

Eigenvectors and eigenvalues are used for example in:

- Vibrational analysis (modal analysis)
  (For example Tacoma Narrows bridge)

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- Many other places

The traditional algorithm

- Via tridiagonal matrices

$$A \xrightarrow{\text{Similarity Transformation}} T \xrightarrow{QR\text{-algorithm}} \Lambda$$

with T a tridiagonal matrix.

The traditional algorithm

- Via tridiagonal matrices

$$A \xrightarrow{\text{Similarity Transformation}} T \xrightarrow{QR\text{-algorithm}} \Lambda$$

with T a tridiagonal matrix.

- Why not

$$A \xrightarrow{QR\text{-algorithm}} \Lambda$$

without using the intermediate tridiagonal matrix.

The traditional algorithm

To find an eigenvalue some steps of the QR-algorithm need to be performed, the complexity of a QR-step applied on

- the symmetric matrix A costs $O(n^3)$ flops;
- the tridiagonal matrix T costs $O(n)$ flops, with n the dimension of the matrix.

Remark that the reduction of A to tridiagonal form is also an order $O(n^3)$ algorithm.

The new algorithm for calculating the eigenvalues

- Via tridiagonal matrices

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with T a tridiagonal matrix.

The new algorithm for calculating the eigenvalues

- Via tridiagonal matrices

$$A \xrightarrow{\text{Similarity Transformation}} T \xrightarrow{QR\text{-algorithm}} \Lambda$$

with T a tridiagonal matrix.

- New approach via semiseparable matrices

$$A \xrightarrow{\text{Similarity Transformation}} S \xrightarrow{QR\text{-algorithm}} \Lambda$$

with S a semiseparable matrix.

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The definition of a semiseparable matrix

Definition

A matrix S is called a lower- (upper-)semiseparable matrix of semiseparability rank 1 if all submatrices which can be taken out of the lower (upper) triangular part of the matrix S have rank ≤ 1.

For example a 5×5 matrix:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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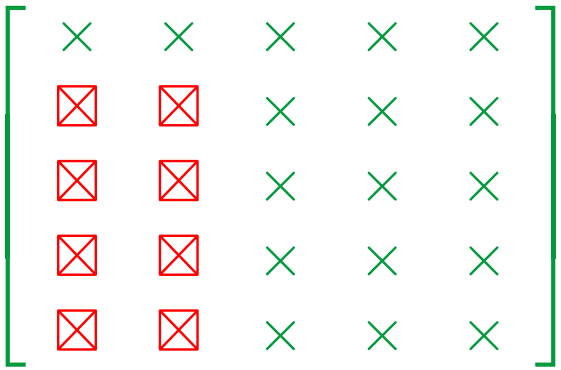

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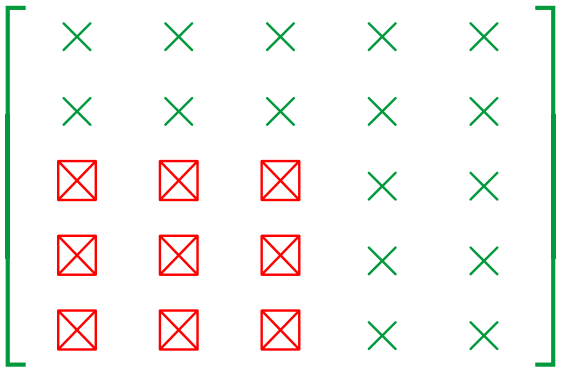

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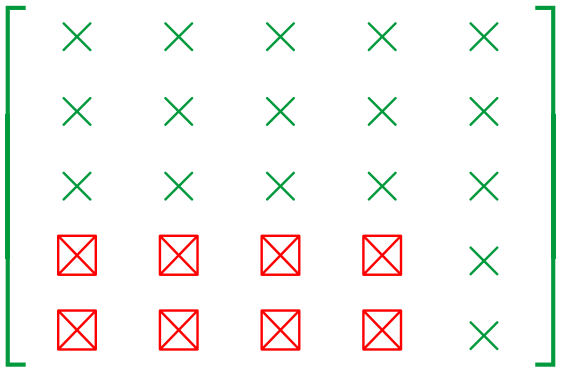

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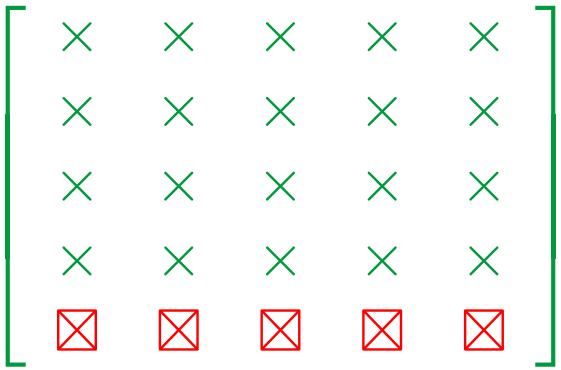

Examples of Semiseparable matrices

- The blocks marked in red are of rank 1

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 2 & 4 & 8 \\ 4 & 4 & 4 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 2 & 4 & 8 \\ 4 & 4 & 4 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 2 & 4 & 8 \\ 4 & 4 & 4 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 2 & 4 & 8 \\ 4 & 4 & 4 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix}$$

Examples of Semiseparable matrices

- The blocks marked in red are of rank 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Semiseparable and tridiagonal matrices

Suppose T is a tridiagonal matrix, then its inverse S:

$$T * S = I = S * T$$

is a semiseparable matrix.

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Suppose T is a tridiagonal matrix, then its inverse S:

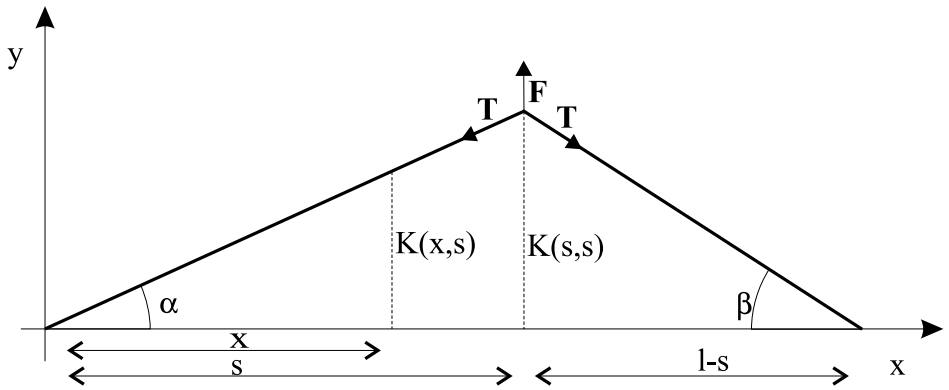
$$T * S = I = S * T$$

is a semiseparable matrix. For example:

$$\frac{1}{5}\begin{bmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{bmatrix} * \begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Examples

- As oscillation matrices



- Covariance matrices in statistics
- Integral equations
- Orthogonal rational functions

Some history

- Oscillation Matrices:
  Gantmacher & Krein: Oscillation matrices, 1937.
  Proved that the inverse of a tridiagonal is a semiseparable matrix (One-pair matrix).
- Statistical analysis:
  Roy & Sarhan, 1956.
- Integral equations:
  Asplund (Father and Son), 1957.

Other people also contributed in the field: Rózsa, Barrett, Feinsilver, Fiedler.

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Symmetric tridiagonal matrices

It is enough to store the diagonal and subdiagonal elements.

For example:

$$\begin{bmatrix} 1 & 7 & & \\ 7 & 2 & 8 & \\ & 8 & 3 & 9 \\ & & 9 & 4 \end{bmatrix}$$

Store:

- diagonal $[1, 2, 3, 4]$
- subdiagonal $[7, 8, 9]$

Symmetric semiseparable matrices
Classical representation

The symmetric semiseparable matrix is represented with two vectors $u = [u_1, u_2, \ldots, u_n]^T$ and $v = [v_1, v_2, \ldots, v_n]^T$.

For example a 5×5 matrix has the following from:

$$\begin{pmatrix} u_1v_1 & & & & \\ u_2v_1 & u_2v_2 & & & \\ u_3v_1 & u_3v_2 & u_3v_3 & & \\ u_4v_1 & u_4v_2 & u_4v_3 & u_4v_4 & \\ u_5v_1 & u_5v_2 & u_5v_3 & u_5v_4 & u_5v_5 \end{pmatrix}$$

Symmetric semiseparable matrices
Classical representation

Example:

$$\begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ 4 & 4 & 4 & \\ 8 & 8 & 8 & 8 \end{bmatrix}$$

The generators:

- $u = [1\ 2\ 4\ 8]$
- $v = [1\ 1\ 1\ 1]$

$$\begin{bmatrix} 1 & & & \\ 0 & 2 & & \\ 0 & 0 & 3 & \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

It is not possible to represent this matrix.

- $u = [1\ 0\ 0\ 0]$
- $v = [1\ ?\ ?\ ?]$

Symmetric semiseparable matrices
Givens-vector representation

The semiseparable matrix is represented with a sequence of Givens transformations and a vector.

$$G = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} \\ s_1 & s_2 & \dots & s_{n-1} \end{bmatrix} \qquad d = [d_1\ d_2\ \dots d_n]$$

For example a 5×5 matrix has the following form:

$$\begin{bmatrix} & d_1 & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

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For example a 5×5 matrix has the following form:

$$\begin{bmatrix} & c_1 d_1 & & & \\ & s_1 d_1 & & d_2 & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

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For example a 5×5 matrix has the following form:

$$\begin{bmatrix} c_1 d_1 & & & \\ c_2 s_1 d_1 & c_2 d_2 & & \\ s_2 s_1 d_1 & s_2 d_2 & d_3 & \\ & & & \end{bmatrix}$$

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Symmetric semiseparable matrices
Givens-vector representation

Example:

$$\begin{bmatrix} 1 & & & \\ 2 & 2 & & \\ 4 & 4 & 4 & \\ 8 & 8 & 8 & 8 \end{bmatrix}$$

Givens-vector representation:

- $d = [9.2195 \quad 9.1652 \quad 8.9443 \quad 8.0000]$
- $G = \begin{bmatrix} 0.1085 & 0.2182 & 0.4472 \\ 0.9941 & 0.9759 & 0.8944 \end{bmatrix}$

$$\begin{bmatrix} 1 & & & \\ 0 & 2 & & \\ 0 & 0 & 3 & \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

- $d = [1\ 2\ 3\ 4]$
- $G = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Conclusion

- Definition
  - Blocks of rank 1: Structured ranks
  - Inverse of a semiseparable matrix
    - Nullity theorem
    - Alternative proof
    - To predict structures of the inverse
    - To predict structures in the QR and LU-decompositions

Conclusion

- Definition
- Representation
  - The classical generator representation
  - The Givens-vector representation
    - The pointwise closure
    - Diagonal matrix with generator representable blocks
  - Diagonal subdiagonal representation
  - Algorithms
    - Retrieving the representation
    - Calculation of the determinant
    - Fast matrix vector multiplication

Conclusion

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- Representation
- Historical overview

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- Historical overview
- Applications
  - Oscillation matrices
  - Covariance matrices
  - Integral equations
  - Orthogonal rational functions

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Transformations used

In the following figures:

- The elements $\times, \otimes$ denote elements in the matrix.
- The elements $\otimes$ will be annihilated by the transformation.
- The elements marked in red will change by the transformation.
- Elements not shown, are assumed to be zero.

Transformations used

To reduce the matrices to tridiagonal or semiseparable form we used two types of transformations:

- Givens transformations:
  - G is a transformation matrix such that

$$G^T \begin{bmatrix} \otimes \\ \times \end{bmatrix} = \begin{bmatrix} 0 \\ \times \end{bmatrix}$$

or

$$\begin{bmatrix} \otimes & \times \end{bmatrix} G = \begin{bmatrix} 0 & \times \end{bmatrix}$$

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or

$$\begin{bmatrix} \otimes & \times \end{bmatrix} G = \begin{bmatrix} 0 & \times \end{bmatrix}$$

  - Embedding gives for example on a 4×4, applying the transformation on the left

$$G^T \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \otimes & \times \\ \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{bmatrix}$$

Transformations used

- Householder transformations:
  - H is a transformation matrix such that

$$H^T \begin{bmatrix} \otimes \\ \vdots \\ \otimes \\ \times \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \times \end{bmatrix}$$

or

$$\begin{bmatrix} \otimes & \cdots & \otimes & \times \end{bmatrix} H = \begin{bmatrix} 0 & \cdots & 0 & \times \end{bmatrix}$$

  - Embedding can be done in an analogous way as for the Givens transformations.

Transformations used

- Similarity transformations on a symmetric matrix A:

$$H^T A H \quad \text{and} \quad G^T A G$$

For example

$$G^T \begin{bmatrix} \times & \times & \otimes \\ \times & \times & \times \\ \otimes & \times & \times \end{bmatrix} G = \begin{bmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

The reduction to tridiagonal form



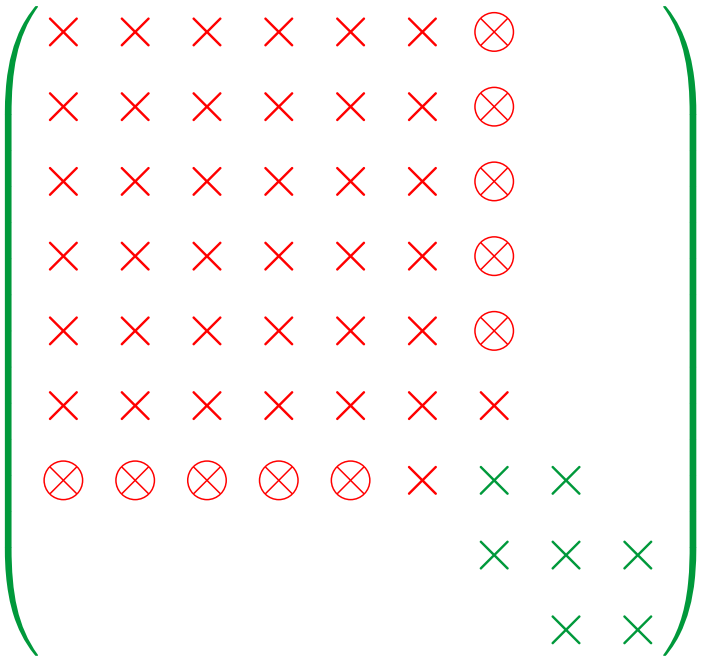
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 8
- Involving row 1 to 8

The reduction to tridiagonal form



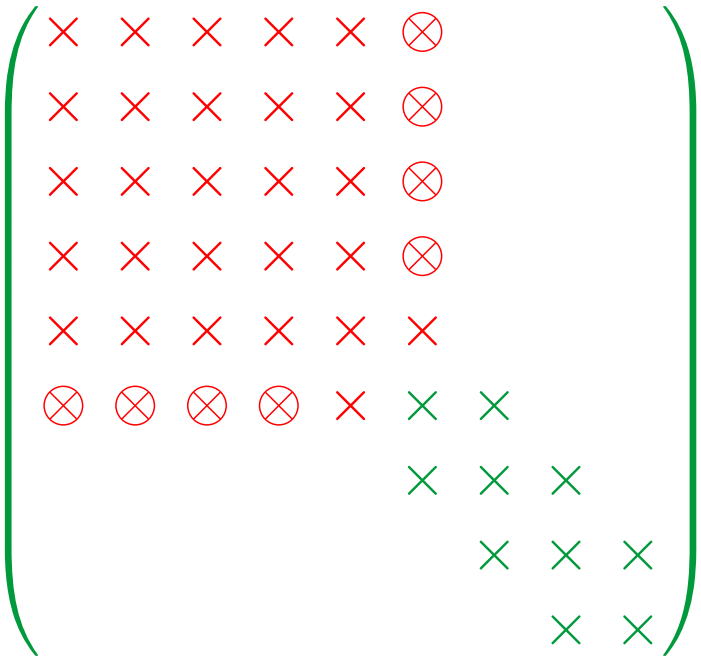
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 7
- Involving row 1 to 7

The reduction to tridiagonal form



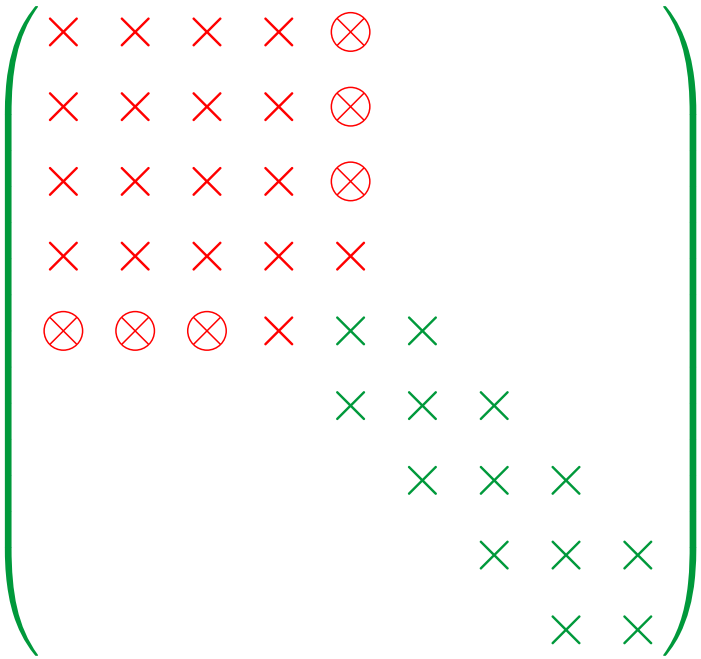
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 6
- Involving row 1 to 6

The reduction to tridiagonal form



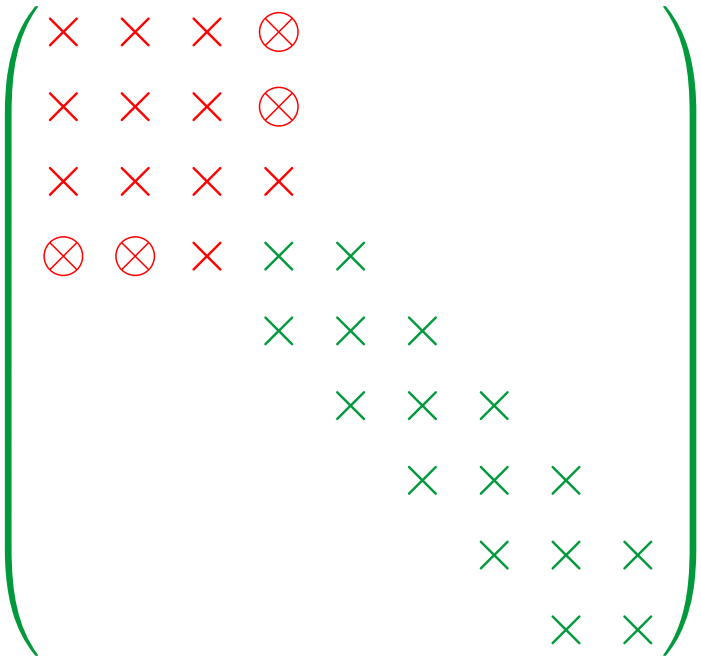
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 5
- Involving row 1 to 5

The reduction to tridiagonal form



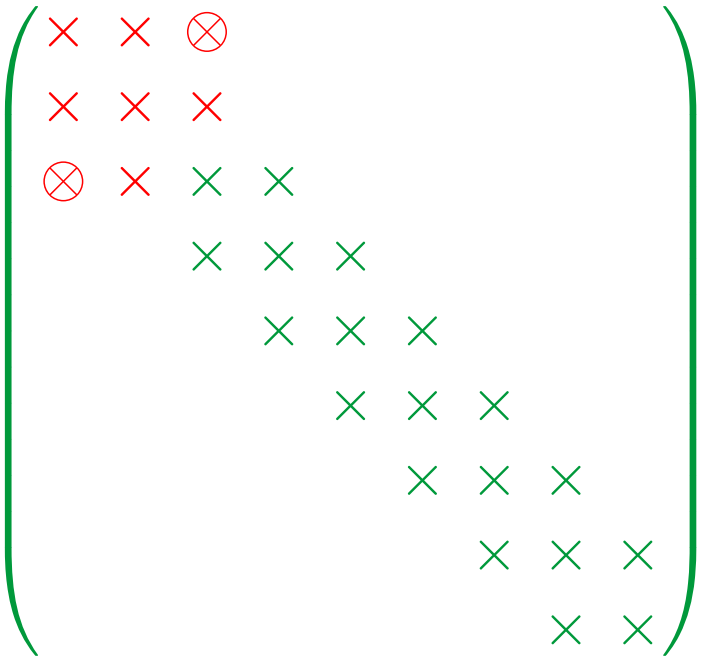
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 4
- Involving row 1 to 4

The reduction to tridiagonal form



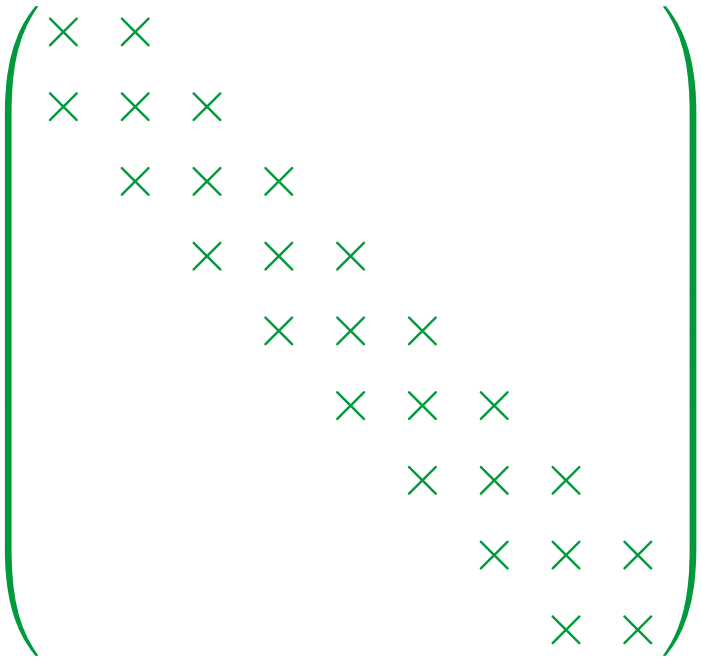
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 3
- Involving row 1 to 3

The reduction to tridiagonal form



- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 2
- Involving row 1 to 2

The reduction to tridiagonal form



- We have now a tridiagonal matrix.

The reduction to semiseparable form

In the following figures:

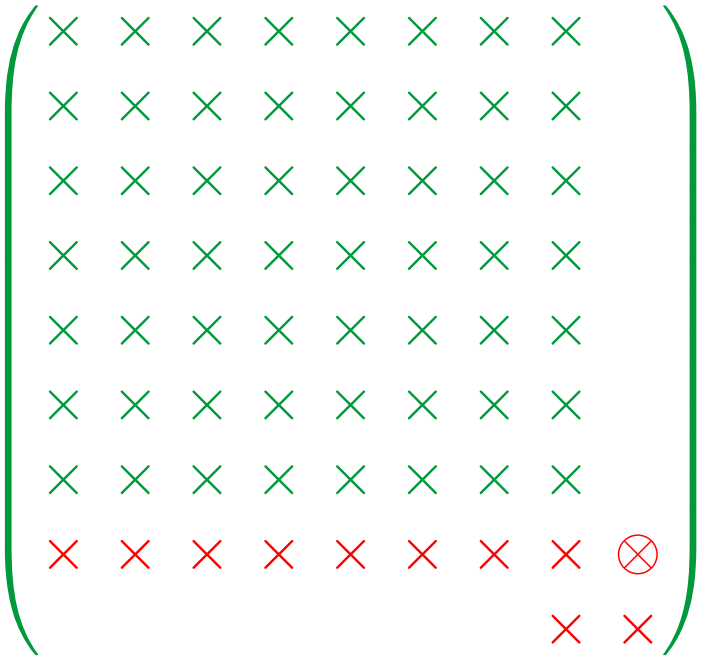
- The same conventions as before.
- The elements ⊠ denote the elements satisfying already the semiseparable structure.

The reduction to semiseparable form



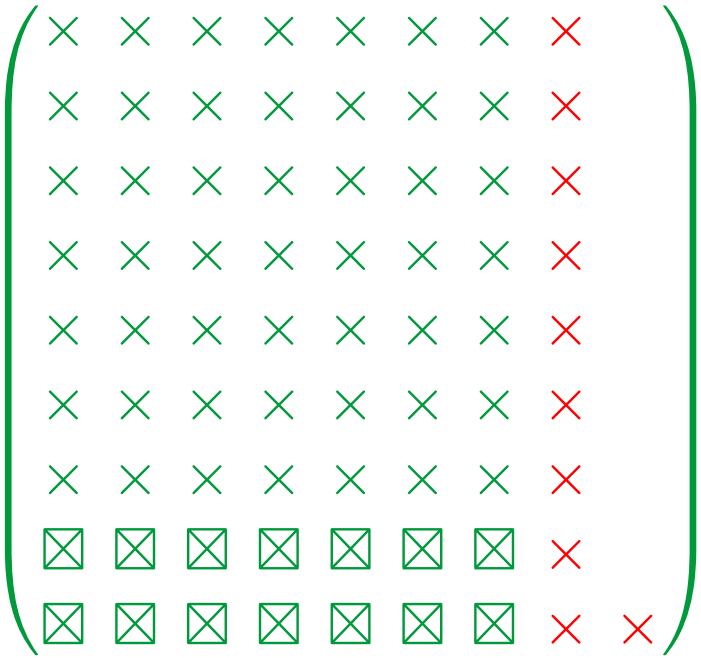
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 8
- Involving row 1 to 8

The reduction to semiseparable form



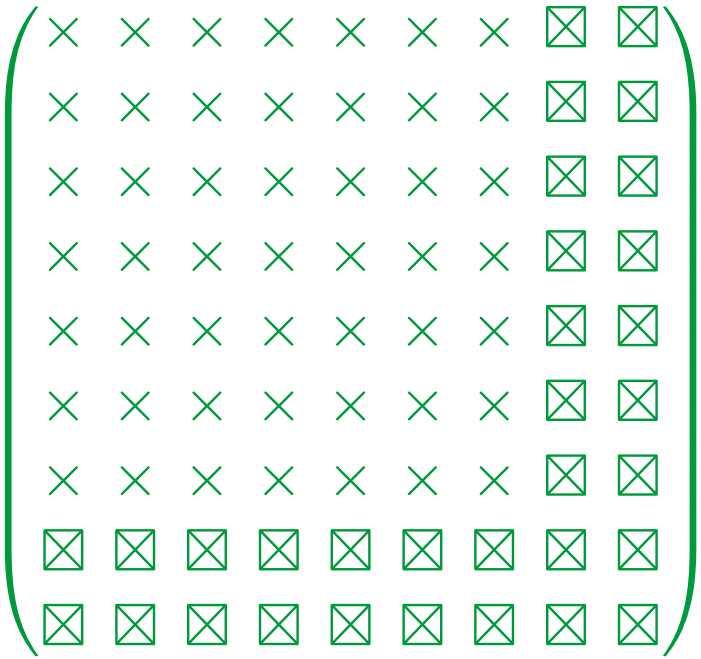
- Apply a Givens transformation on the left of the matrix.
- Involving row 8 and 9

The reduction to semiseparable form



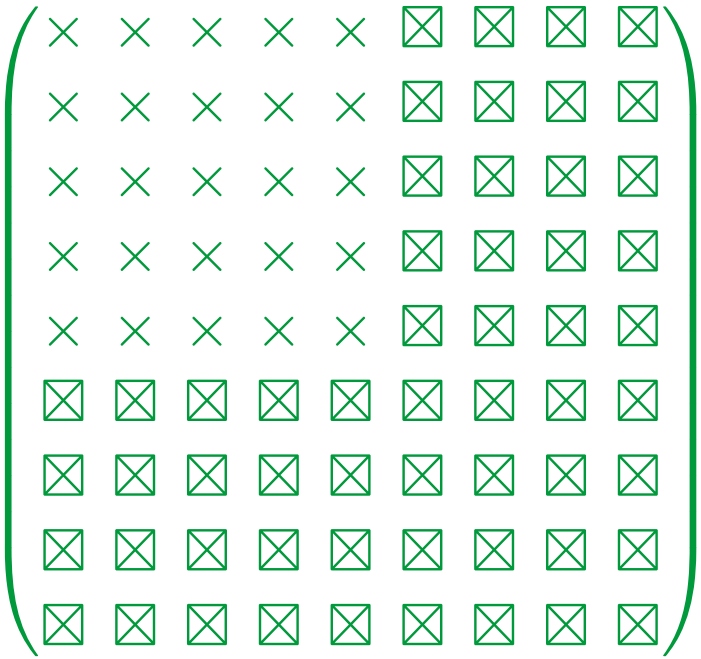
- Apply a Givens transformation of to the right of the matrix.
- Involving column 8 and 9

The reduction to semiseparable form



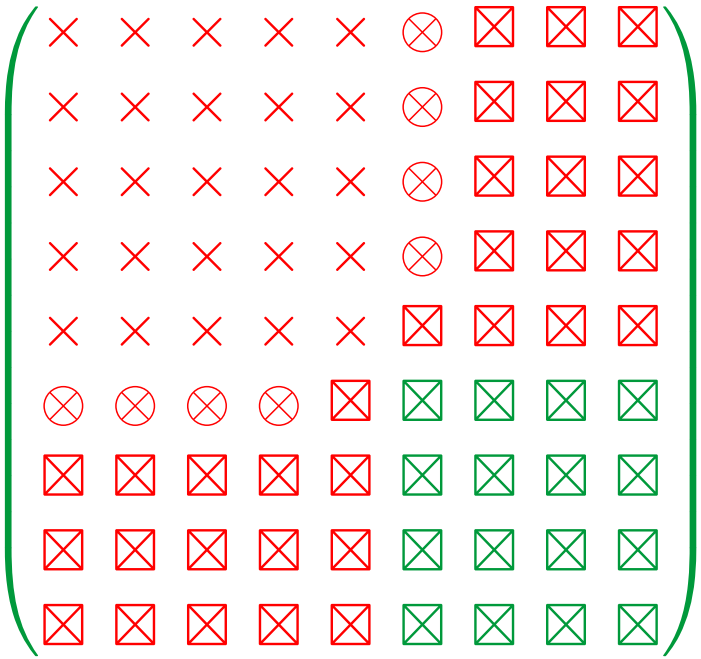
- Already two rows satisfy the semiseparable structure.

The reduction to semiseparable form



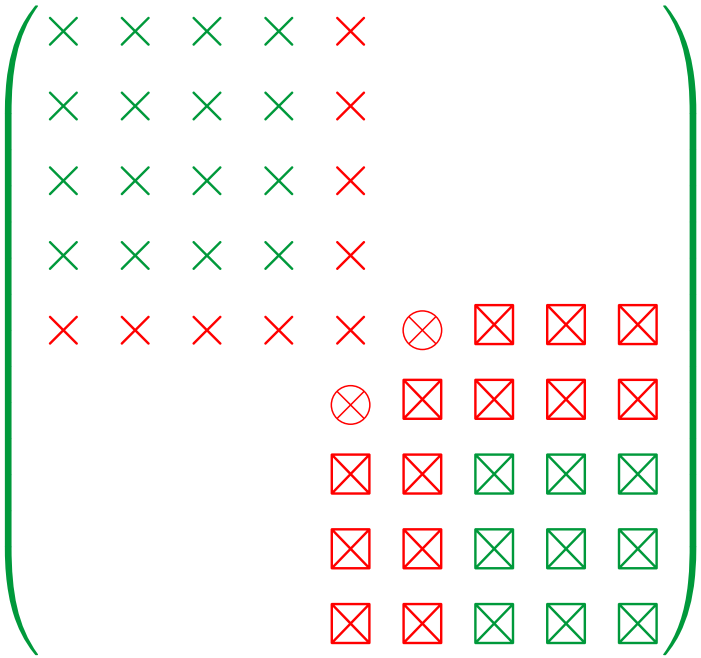
- Suppose we have now 4 rows and columns satisfying the semiseparable structure. We will show now how to add one row and column to the semiseparable structure.

The reduction to semiseparable form



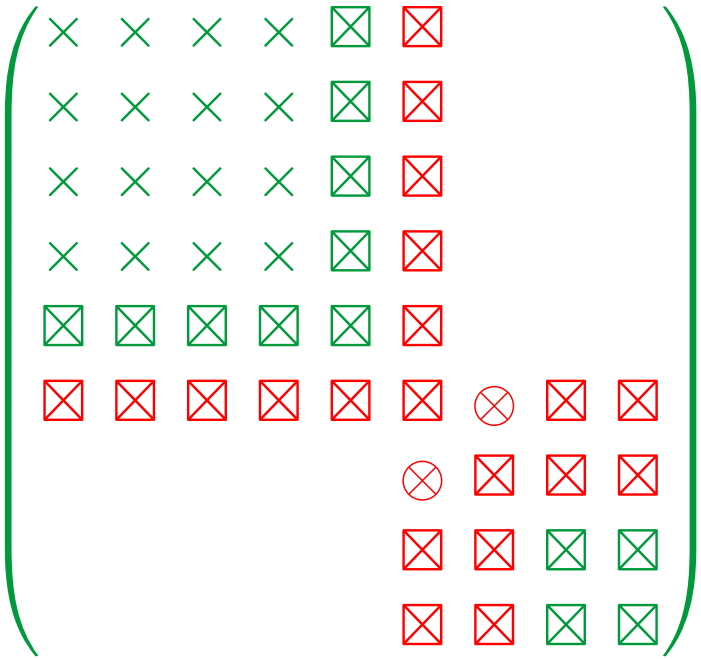
- Apply a Householder transformation to both sides of the matrix.
- Involving column 1 to 5
- Involving row 1 to 5

The reduction to semiseparable form



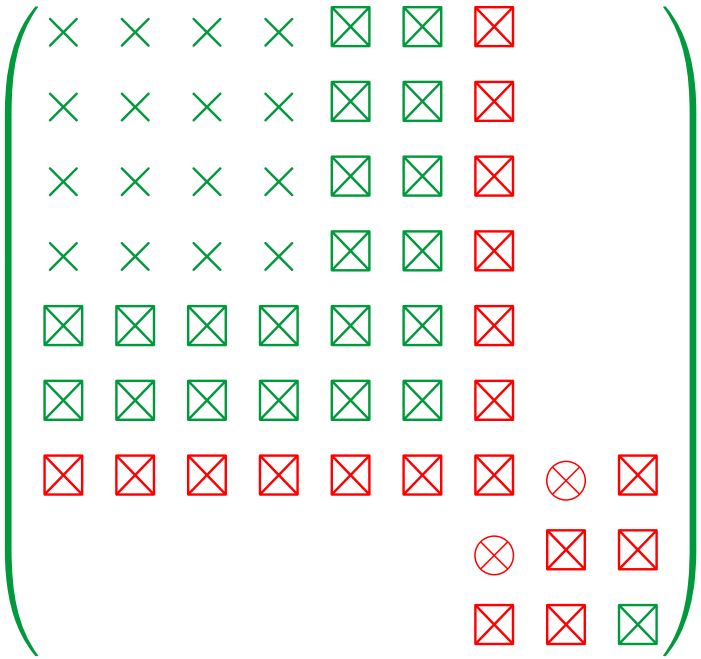
- Apply a Givens transformation to both sides of the matrix.
- Involving column 5 and 6
- Involving row 5 and 6

The reduction to semiseparable form



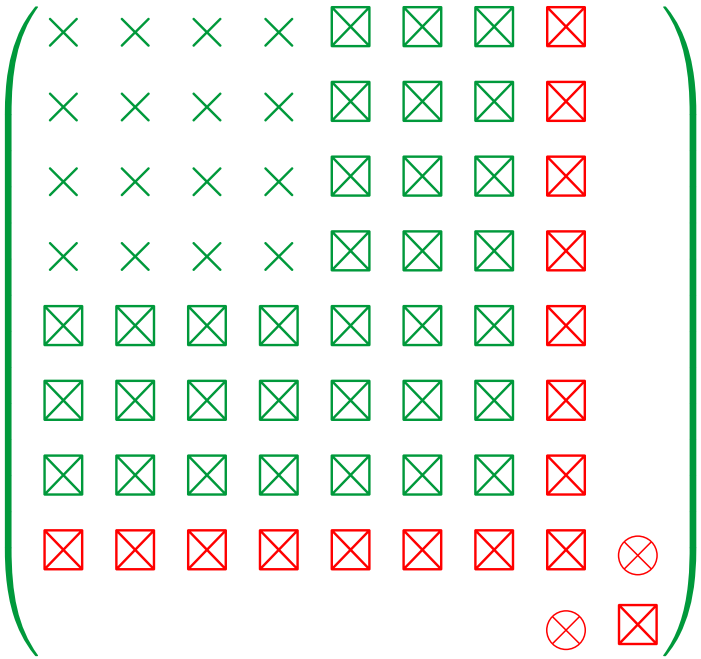
- Apply a Givens transformation to both sides of the matrix.
- Involving column 6 and 7
- Involving row 6 and 7

The reduction to semiseparable form



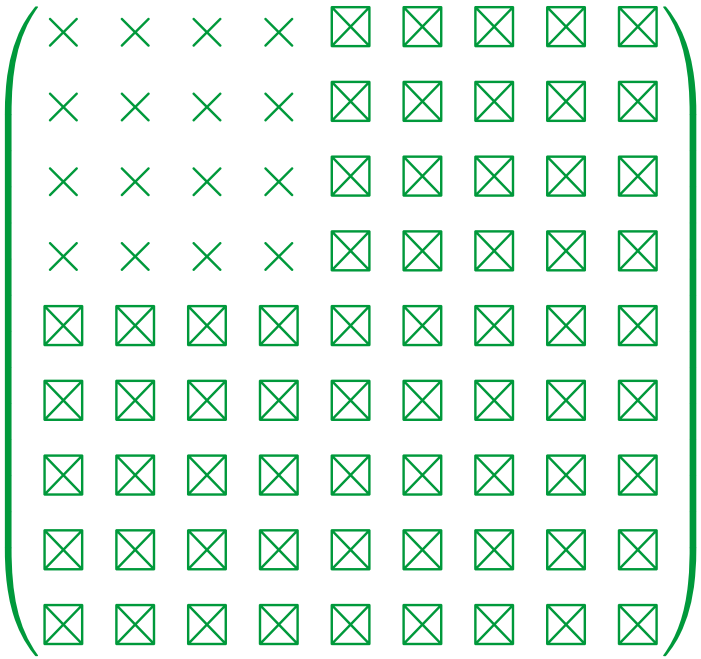
- Apply a Givens transformation to both sides of the matrix.
- Involving column 7 and 8
- Involving row 7 and 8

The reduction to semiseparable form



- Apply a Givens transformation to both sides of the matrix.
- Involving column 8 and 9
- Involving row 8 and 9

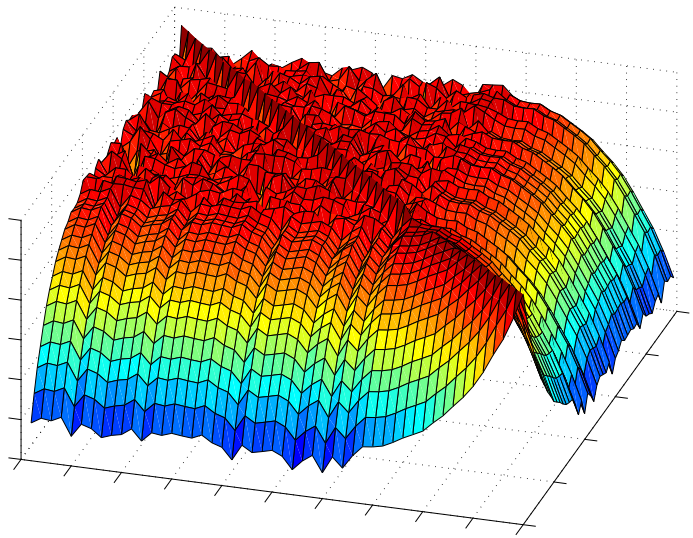
The reduction to semiseparable form

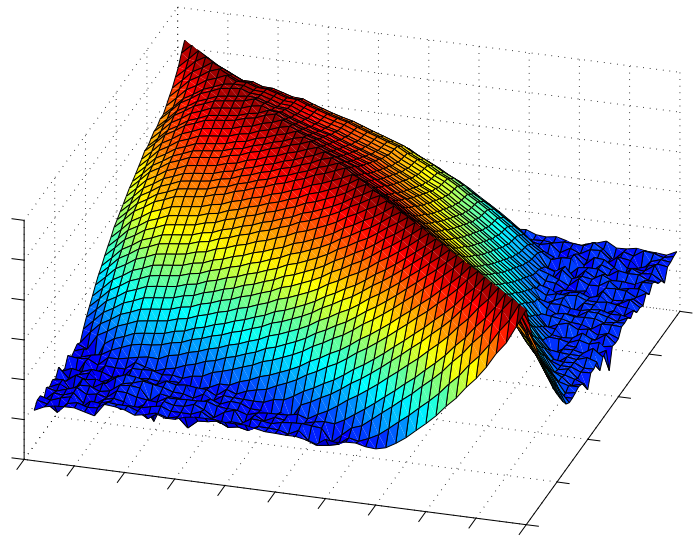


- The resulting matrix has now the last 5 rows and 5 columns satisfying the semiseparable structure.

Example of the reduction algorithm

In the pictures a surface plot of the different steps in the reduction algorithm are shown.





Comparison in complexity

The reduction algorithms have the following complexities

- From symmetric to tridiagonal form
  Costs:
  - $4n^3/3 + O(n^2)$ with Householders,
  - $2n^3 + O(n^2)$ with Givens transformations .
- From symmetric to semiseparable form
  Costs:
  - The same as above plus an extra $9n^2 + O(n)$.

Outline

Introduction
- Eigenvalue problem
- Semiseparable matrices
- Representation

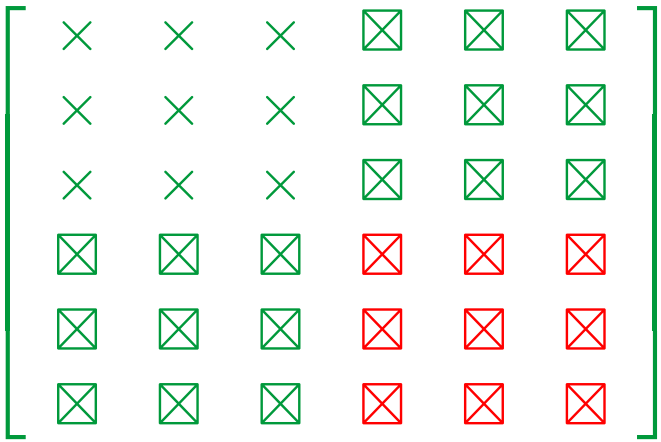
The reduction to semiseparable form
- The reduction algorithms
- The convergence properties

QR-algorithms for semiseparable matrices
- The implicit and the explicit approach
- The implicit QR-algorithms
- Complete algorithm via semiseparable matrices

Future work

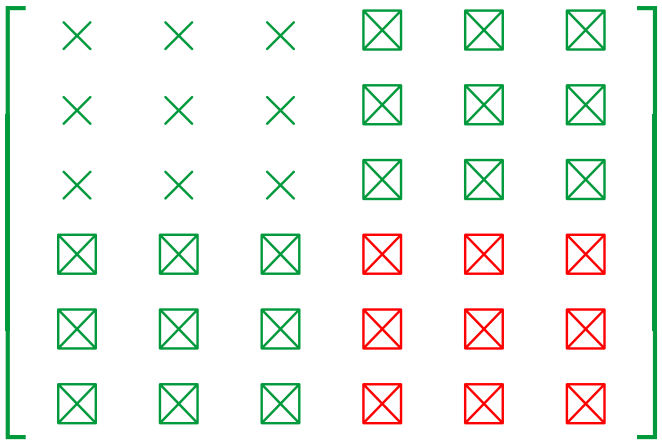
The two properties

Intermediate matrices in the reduction algorithm are of the following form:



The two properties

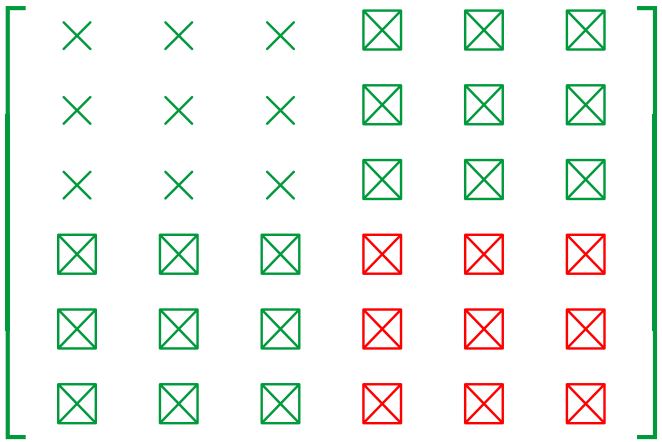
Intermediate matrices in the reduction algorithm are of the following form:



- The red part contains as eigenvalues the Lanczos Ritz values (similar as in the tridiagonal case).

The two properties

Intermediate matrices in the reduction algorithm are of the following form:



- The red part contains as eigenvalues the Lanczos Ritz values (similar as in the tridiagonal case).
- On the red part one step of subspace iteration is performed.

Examples of Lanczos Convergence

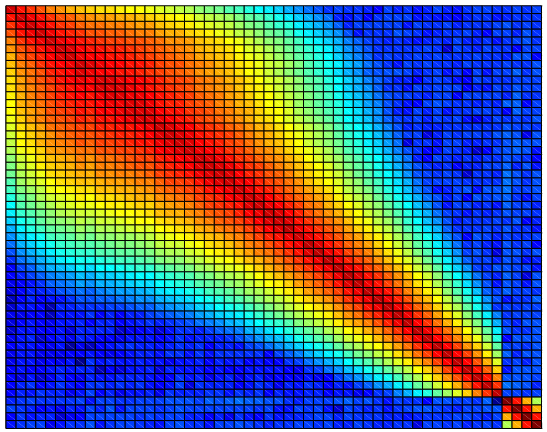
The block in the lower right part contains the eigenvalues with respect to the Krylov subspace:

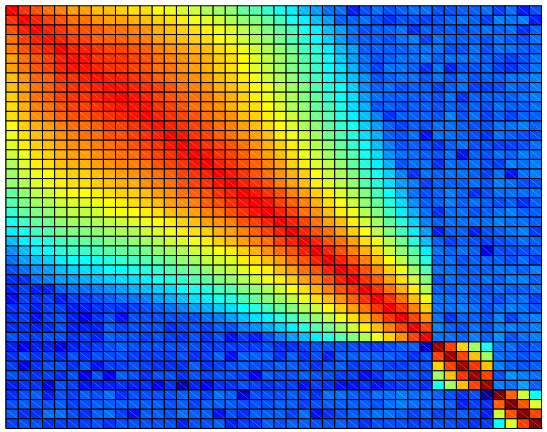
$$\mathcal{K}_k(A, e_1) = \langle e_1, Ae_1, \ldots, A^{k-1}e_1 \rangle.$$

These so-called Lanczos-Ritz values approximate the eigenvalues of the original matrix A.

Examples of the subspace convergence

When there are clusters of eigenvalues, the subspace iteration will create blocks in the matrix.
For example eigenvalues $[1:50, 1000:1004]$ and $[1:35, 1000:1004, 10000:10004]$ will give a semiseparable matrix of the following form





Conclusions

- Reduction Algorithms
  - Similarity reduction of a symmetric to a semiseparable matrix
  - Similarity reduction of a nonsymmetric to a Hessenberg-like matrix
  - Orthogonal reduction of a matrix to an upper triangular semiseparable form

Conclusions

- Reduction Algorithms
- Convergence properties
  - General framework for the Lanczos convergence
  - Theoretical proofs of all the convergence behaviors
  - Interaction analysis of the two convergence behaviors

Outline

- Introduction
  - Eigenvalue problem
  - Semiseparable matrices
  - Representation
- The reduction to semiseparable form
  - The reduction algorithms
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- QR-algorithms for semiseparable matrices
  - The implicit and the explicit approach
  - The implicit QR-algorithms
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- Future work

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The QR-algorithm, the explicit approach

The QR-decomposition of a matrix A:

$$A = QR.$$

For which

- Q is an orthogonal matrix;
- R is an upper triangular matrix.

The QR-algorithm, the explicit approach

Suppose we have a shift κ, and we calculate the QR-decomposition of the matrix $A - \kappa I$:

$$A - \kappa I = QR$$

and let us denote with $\tilde{A}$ the matrix:

$$\tilde{A} = RQ + \kappa I.$$

The matrix $\tilde{A}$ is the result of performing one step of the QR-algorithm on the matrix A.

When we perform the operations as presented here, this is called the explicit approach.

The implicit approach

We can rewrite the formulas from before as:

$$Q^{-1}(A - \kappa I) = R$$

substituting R in the next equation

$$\tilde{A} = RQ + \kappa I$$

gives us

$$\tilde{A} = Q^{-1}AQ.$$

When we apply the transformations Q and Q^{-1} at the same time, we have an implicit algorithm.

Why an implicit algorithm?

- If only the eigenvalues are desired, we do not need to store Q.
- We do not lose accuracy due to the adding and subtracting of the shift matrix κI.

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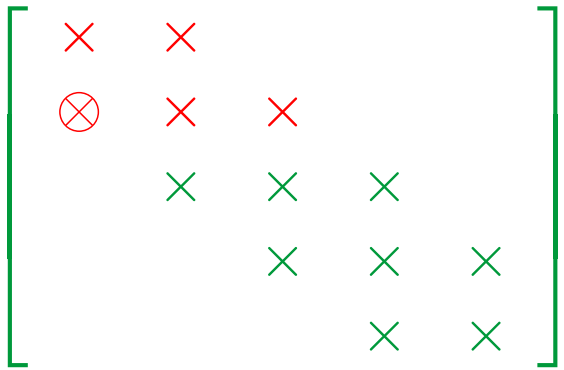
QR-factorization of tridiagonal matrices

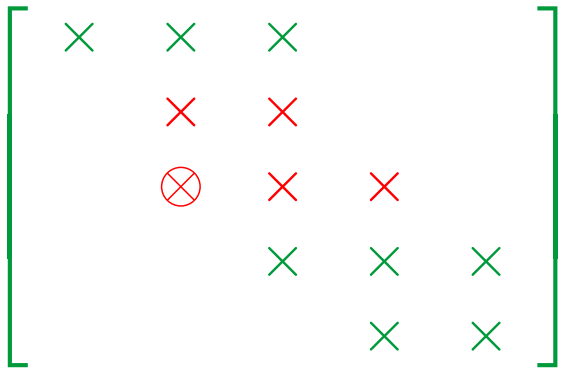
The QR-factorization of the tridiagonal matrix $A - \kappa I$, consists of $n - 1$ Givens transformations



- Apply a Givens transformation on the left.
- Involving row 1 and 2.

QR-factorization of tridiagonal matrices

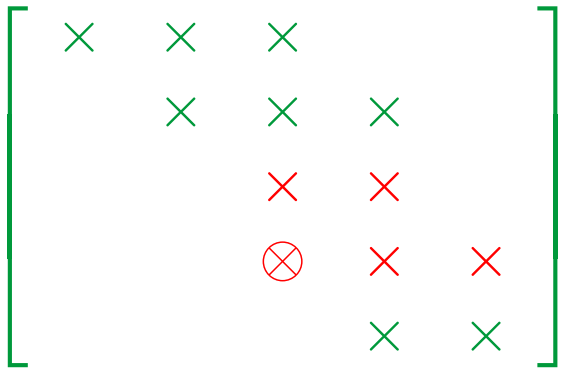
The QR-factorization of the tridiagonal matrix $A - \kappa I$, consists of $n - 1$ Givens transformations



- Apply a Givens transformation on the left.
- Involving row 2 and 3.

QR-factorization of tridiagonal matrices

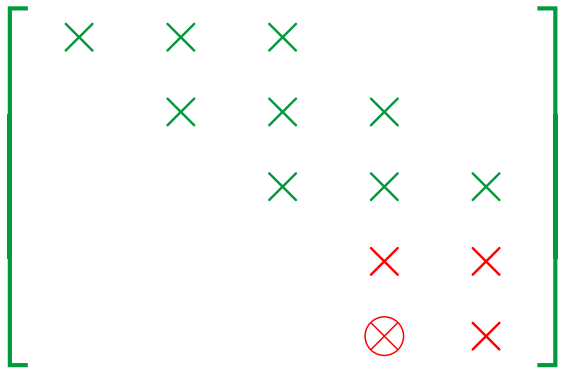
The QR-factorization of the tridiagonal matrix $A - \kappa I$, consists of $n - 1$ Givens transformations



- Apply a Givens transformation on the left.
- Involving row 3 and 4.

QR-factorization of tridiagonal matrices

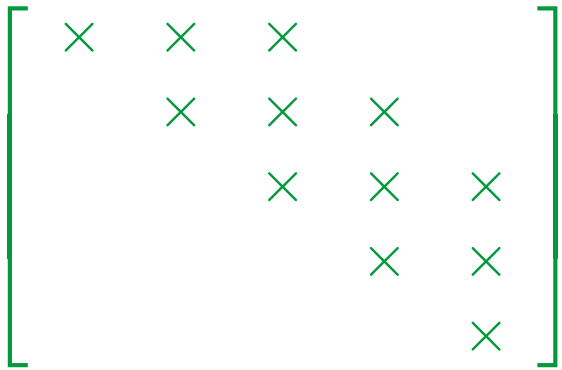
The QR-factorization of the tridiagonal matrix $A - \kappa I$, consists of $n - 1$ Givens transformations



- Apply a Givens transformation on the left.
- Involving row 4 and 5.

QR-factorization of tridiagonal matrices

The QR-factorization of the tridiagonal matrix $A - \kappa I$, consists of $n - 1$ Givens transformations



- We have now an upper triangular matrix R.

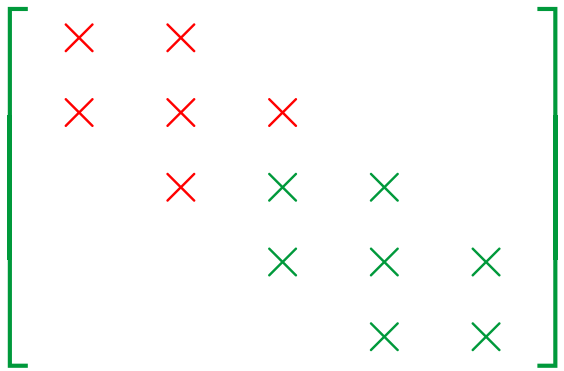
QR-factorization of tridiagonal matrices

Completing the QR-step by applying these transformations on the right gives us:

- Again a tridiagonal matrix.

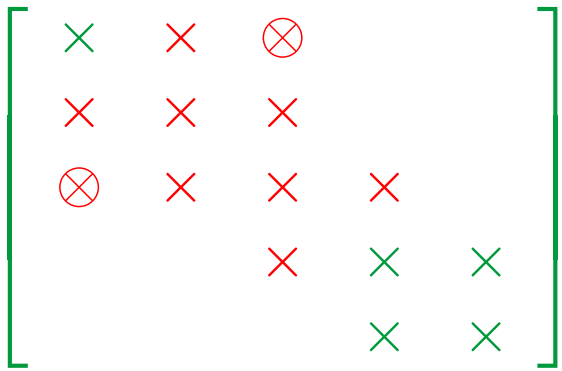
Implicit QR-algorithm for tridiagonal matrices

As mentioned before we want to apply the transformations at both sides at the same time



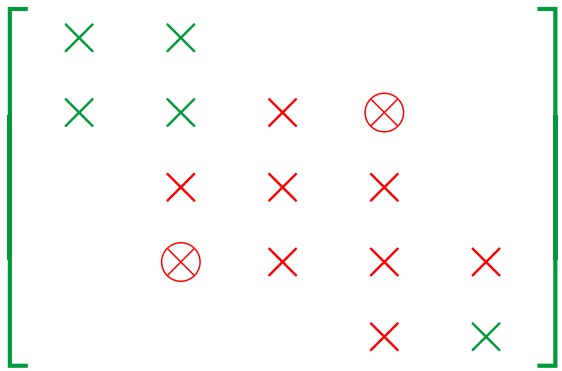
- Apply the first Givens transformation on both sides of the matrix.
- Involving row 1 and 2.
- Involving column 1 and 2.

Implicit QR-algorithm for tridiagonal matrices



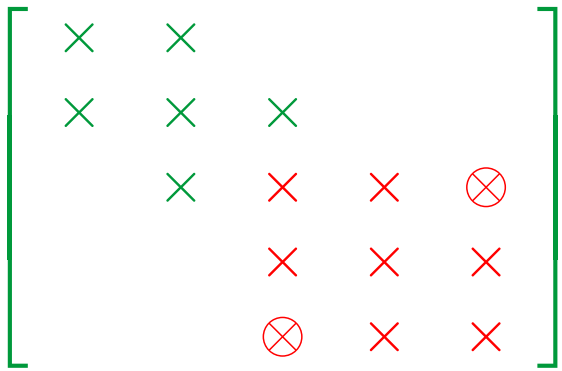
- Apply a Givens transformation to annihilate the bulge.
- Involving row 2 and 3.
- Involving column 2 and 3.

Implicit QR-algorithm for tridiagonal matrices



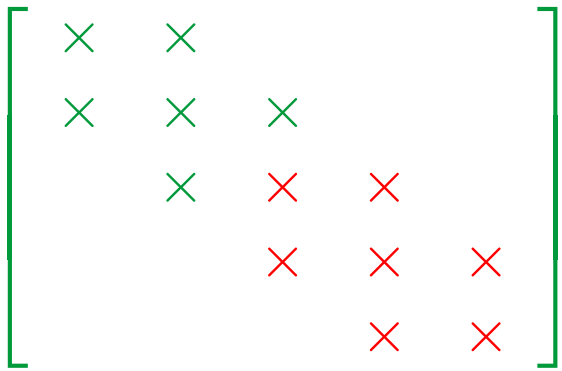
- Apply a Givens transformation to annihilate the bulge.
- Involving row 3 and 4.
- Involving column 3 and 4.

Implicit QR-algorithm for tridiagonal matrices



- Apply a Givens transformation to annihilate the bulge.
- Involving row 4 and 5.
- Involving column 4 and 5.

Implicit QR-algorithm for tridiagonal matrices



- We have again a tridiagonal matrix.

Implicit QR-algorithm for tridiagonal matrices

For an implicit QR-algorithm we need

- QR-factorization.
- Proof that one step of QR maintains the structure
- Chasing technique
- Proof that the implicit and the explicit resulting matrices are equal, this is done by the implicit Q-theorem.

QR-factorization for semiseparable matrices

This is the work of Van Camp, Mastronardi and Van Barel. We have to perform $2n-2$ Givens transformations on the matrix $S-\kappa I$.

- The first $n-1$ Givens transformations Q_1 give us

$$Q_1(S-\kappa I) = \tilde{H}$$

and $\tilde{H}$ is a Hessenberg matrix.

- The second $n-1$ Givens transformations Q_2 give us

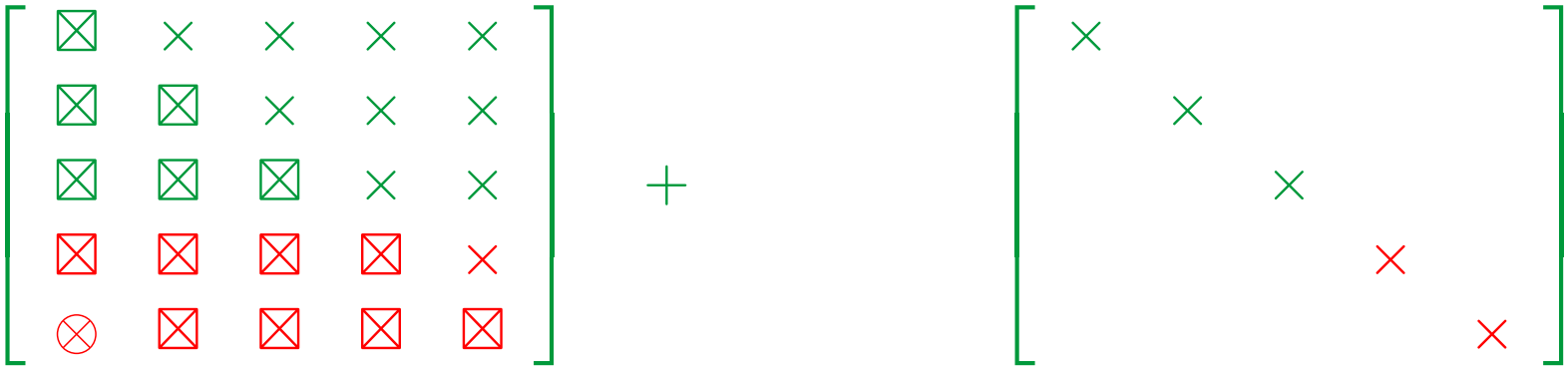
$$Q_2\tilde{H} = \tilde{R}$$

and R is a new upper triangular matrix.

QR-factorization for semiseparable matrices

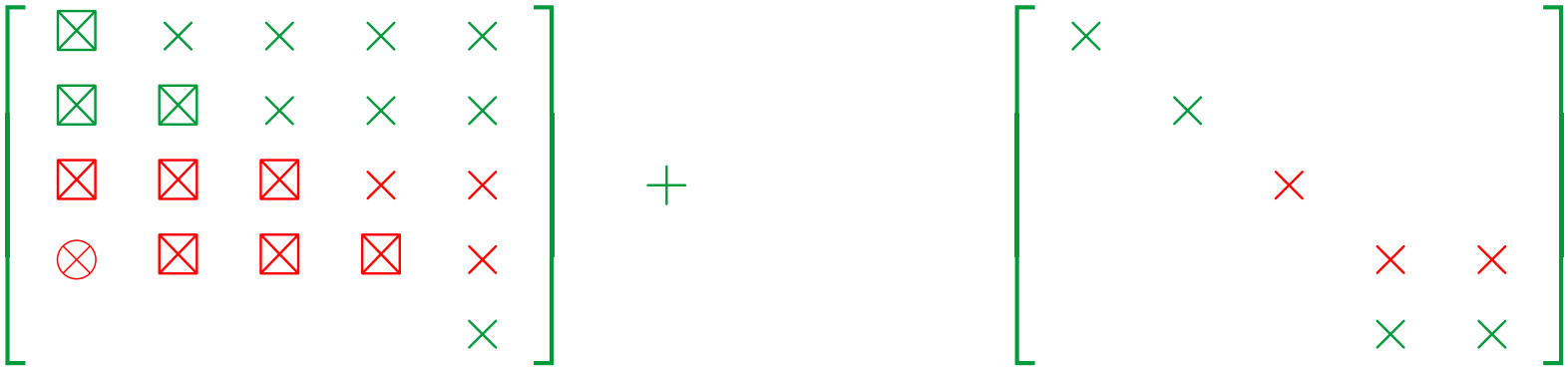
Let us illustrate this:

Apply a Givens transformation on row 4 and 5.



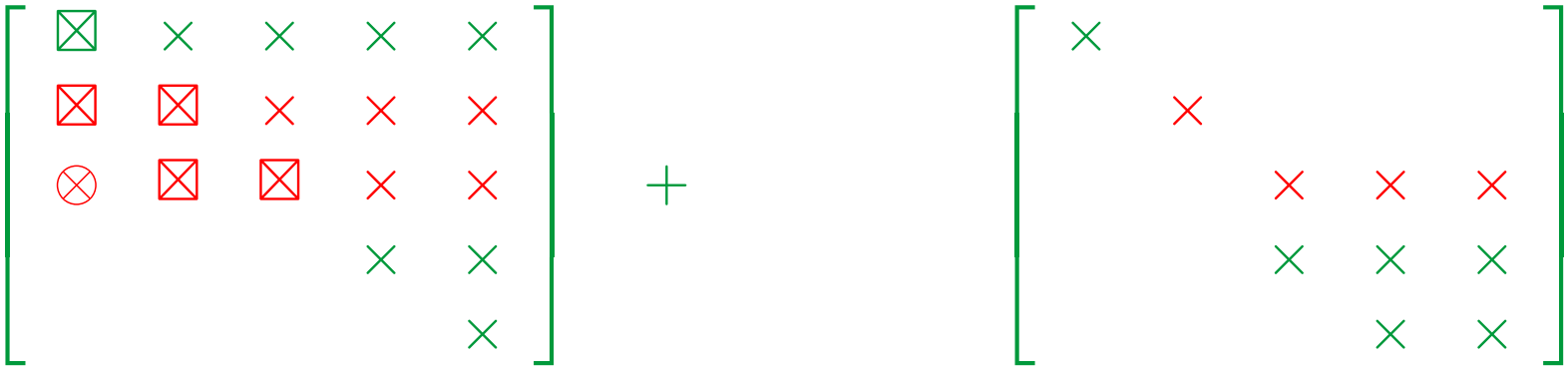
QR-factorization for semiseparable matrices

Apply a Givens transformation on row 3 and 4



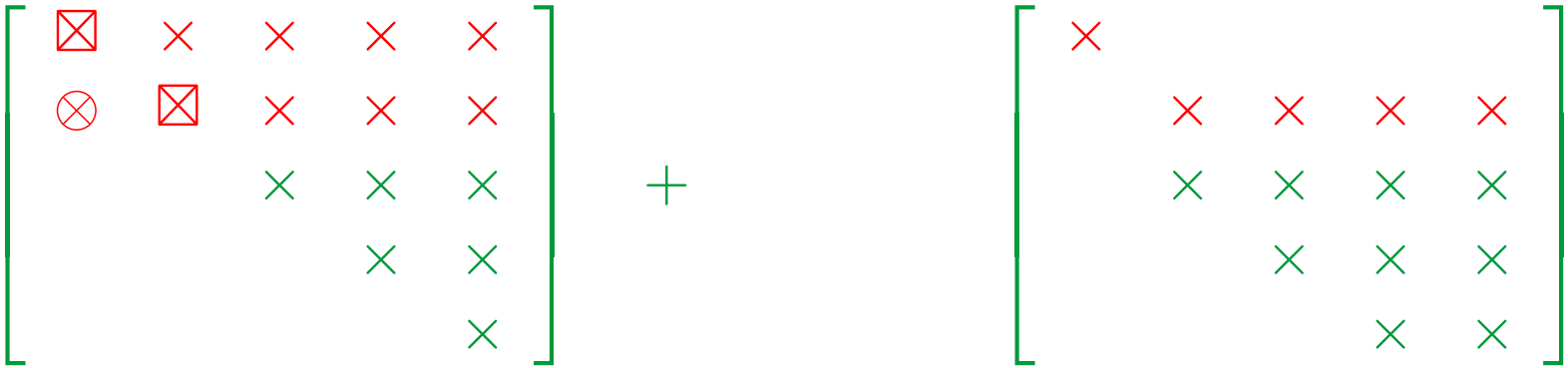
QR-factorization for semiseparable matrices

Apply a Givens transformation on row 2 and 3.



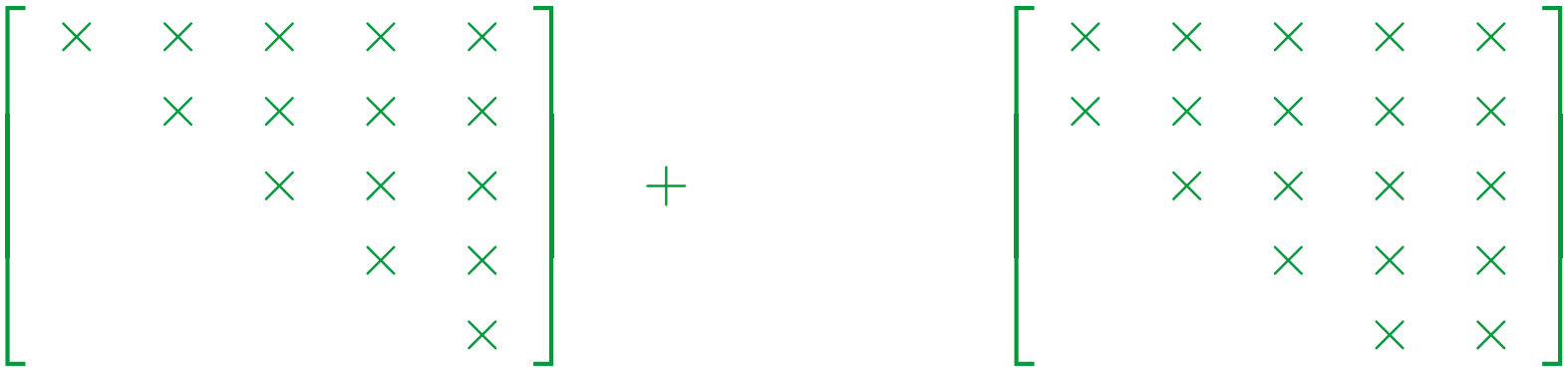
QR-factorization for semiseparable matrices

Apply a Givens transformation on row 1 and 2.



QR-factorization for semiseparable matrices

A summation of both of the matrices gives us a Hessenberg matrix.



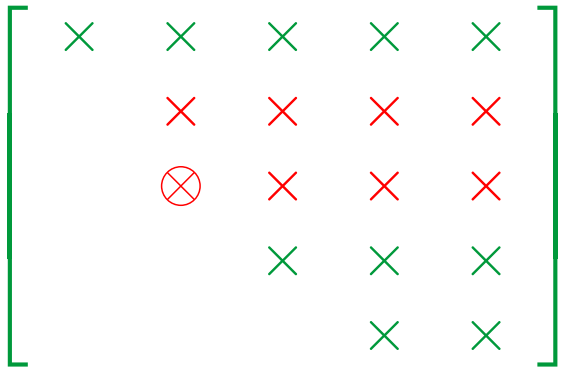
QR-factorization for semiseparable matrices

The QR-factorization of the Hessenberg matrix is similar as the one of a tridiagonal matrix.



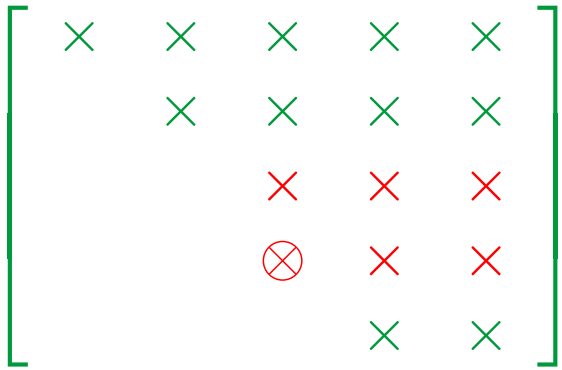
- Apply a Givens transformation on the left.
- Involving row 1 and 2.

QR-factorization for semiseparable matrices



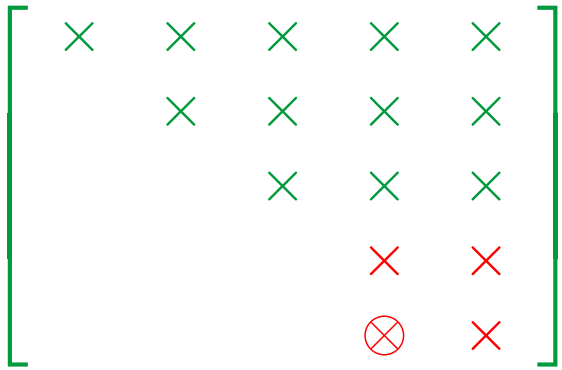
- Apply a Givens transformation on the left.
- Involving row 2 and 3.

QR-factorization for semiseparable matrices



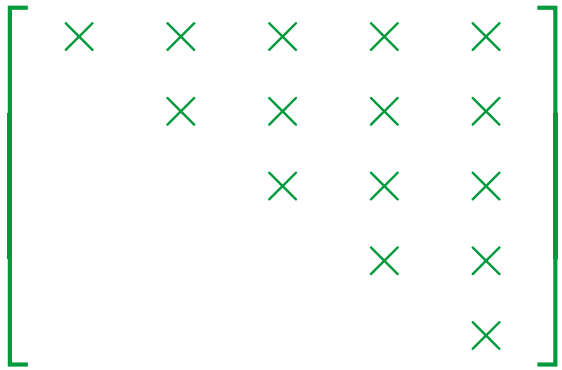
- Apply a Givens transformation on the left.
- Involving row 3 and 4.

QR-factorization for semiseparable matrices



- Apply a Givens transformation on the left.
- Involving row 4 and 5.

QR-factorization for semiseparable matrices



- We have now an upper triangular matrix R.

QR-factorization for semiseparable matrices

Completing the QR-step by applying these transformations on the right gives us:

- Again a semiseparable matrix.

QR-factorization for semiseparable matrices

For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
▶ QR maintains the tridiagonal structure	
▶ Chase technique	
▶ Proof of correctness	

QR-factorization for semiseparable matrices

For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
▶ QR maintains the tridiagonal structure	▶ QR maintains the semiseparable structure
▶ Chase technique	
▶ Proof of correctness	

Implicit QR-algorithm for semiseparable matrices

We want an implicit QR-algorithm for semiseparable matrices and therefore, we want to calculate the following matrix:

$$\tilde{S} = Q_2^{-1} Q_1^{-1} S Q_1 Q_2$$

- The first transformation Q_1 is rather easy, namely

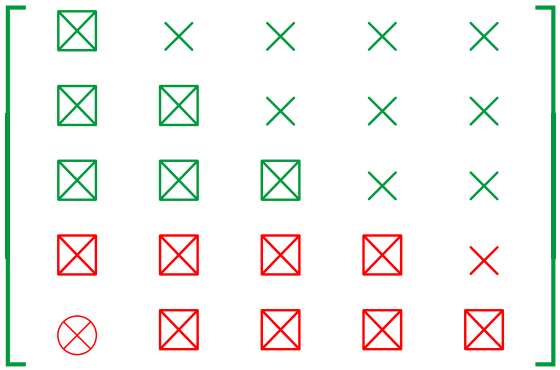
$$Q_1^{-1} S Q_1.$$

- For the second transformation we need a chasing technique.

$$Q_2^{-1} Q_1^{-1} S Q_1 Q_2$$

Implicit QR-algorithm for semiseparable matrices

Applying the first sequence of Givens transformations



- Apply a Givens transformation on the left of the matrix.
- Involving row 4 and 5.

Implicit QR-algorithm for semiseparable matrices

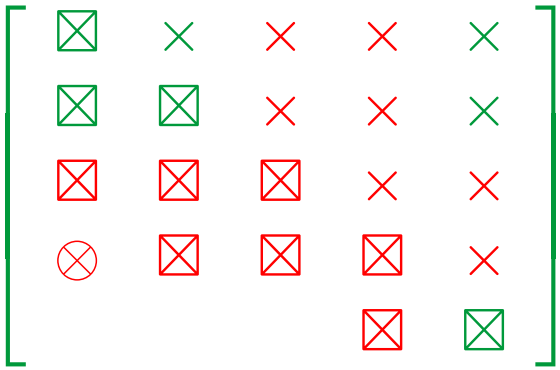
Applying the first sequence of Givens transformations



- Complete the similarity transformation, apply the Givens transformation to the right.
- Involving column 4 and 5.

Implicit QR-algorithm for semiseparable matrices

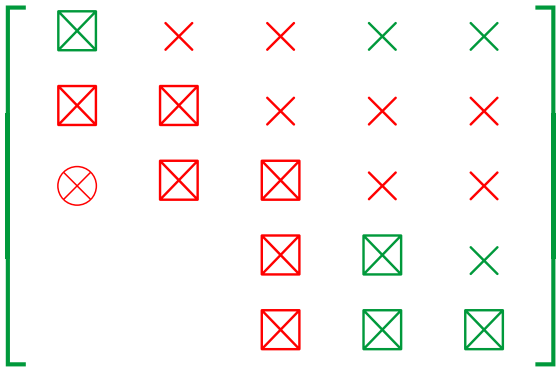
Applying the first sequence of Givens transformations



- Apply a Givens transformation on both sides of the matrix.
- Involving row 3 and 4.
- Involving column 3 and 4.

Implicit QR-algorithm for semiseparable matrices

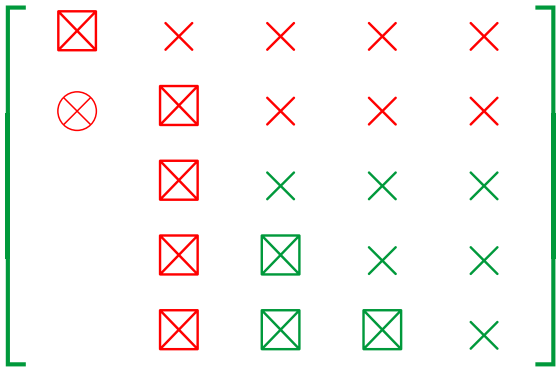
Applying the first sequence of Givens transformations



- Apply a Givens transformation on both sides of the matrix.
- Involving row 2 and 3.
- Involving column 2 and 3.

Implicit QR-algorithm for semiseparable matrices

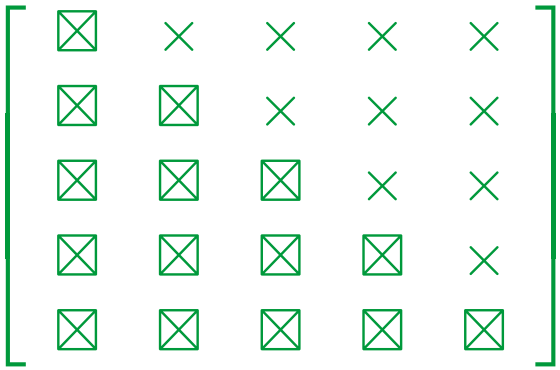
Applying the first sequence of Givens transformations



- Apply a Givens transformation on both sides of the matrix.
- Involving row 1 and 2.
- Involving column 1 and 2.

Implicit QR-algorithm for semiseparable matrices

Applying the first sequence of Givens transformations



- And we have again a semiseparable matrix

Implicit QR-algorithm for semiseparable matrices

For the following sequence of transformations we use a Givens transformation G such that.

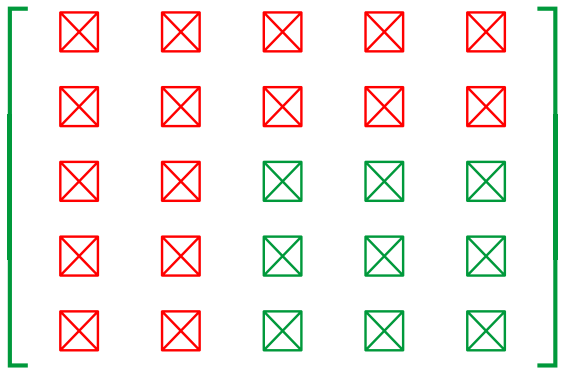
$$G^T \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} G = \begin{bmatrix} \boxtimes & \times & \times \\ \boxtimes & \boxtimes & \times \\ \boxtimes & \boxtimes & \boxtimes \end{bmatrix}$$

Implicit QR-algorithm for semiseparable matrices

In the following figures:

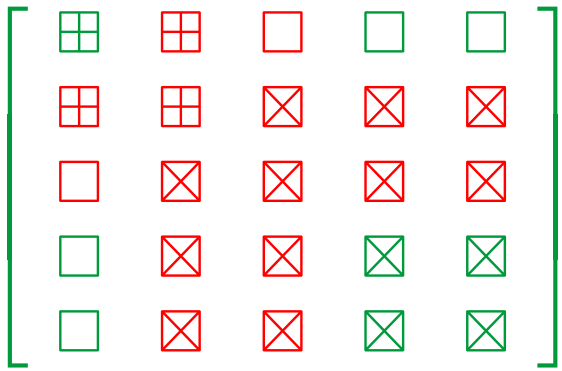
- The elements ⊠ the elements satisfying already the semiseparable structure which we will chase away.
- The elements ⊞ denote the elements of the new semiseparable matrix.
- The elements □ denote the elements of the remaining rank 1 blocks.

Implicit QR-algorithm for semiseparable matrices



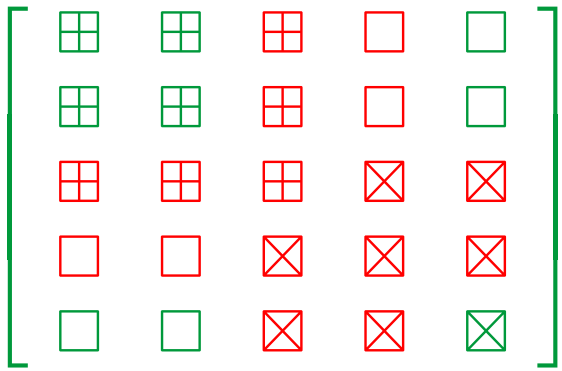
- Apply the first Givens transformation on both sides of the matrix.
- Involving Column 1 and 2.
- Involving Row 1 and 2.

Implicit QR-algorithm for semiseparable matrices



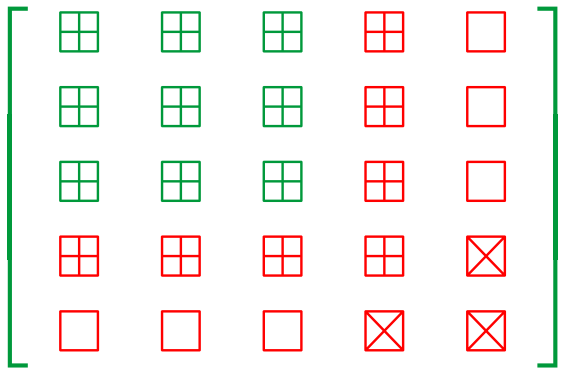
- Apply the first Givens transformation on both sides of the matrix.
- Involving Column 2 and 3.
- Involving Row 2 and 3.

Implicit QR-algorithm for semiseparable matrices



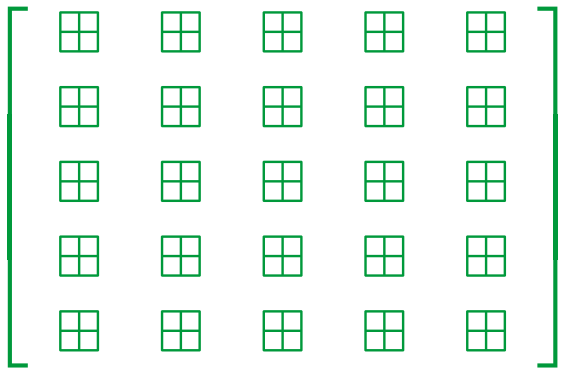
- Apply the first Givens transformation on both sides of the matrix.
- Involving Column 3 and 4.
- Involving Row 3 and 4.

Implicit QR-algorithm for semiseparable matrices



- Apply the first Givens transformation on both sides of the matrix.
- Involving Column 4 and 5.
- Involving Row 4 and 5.

Implicit QR-algorithm for semiseparable matrices



- The resulting matrix is again semiseparable.

Implicit QR-algorithm for semiseparable matrices

For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
▶ QR maintains the tridiagonal structure	▶ QR maintains the semiseparable structure
▶ Chasing technique	
▶ Proof of correctness	

Implicit QR-algorithm for semiseparable matrices

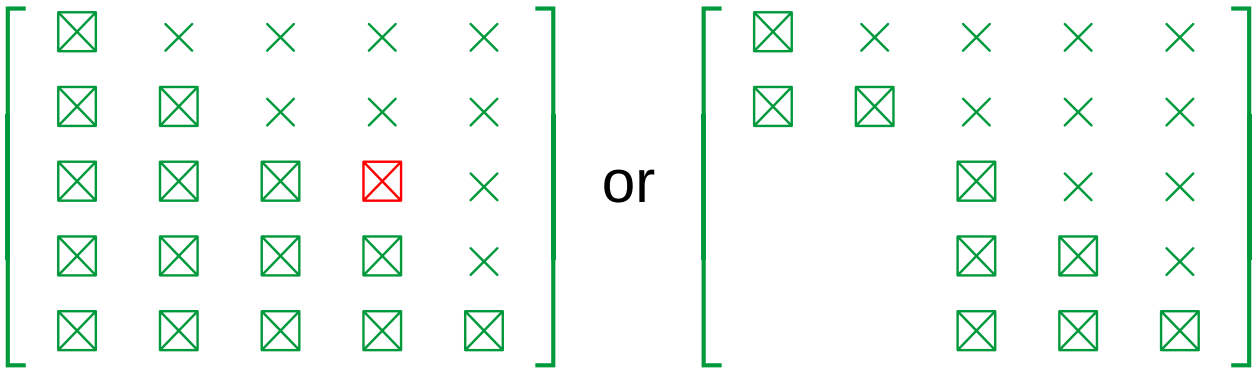
For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
▶ QR maintains the tridiagonal structure	▶ QR maintains the semiseparable structure
▶ Chasing technique	▶ Chasing technique
▶ Proof of correctness	

The implicit Q-theorem for semiseparable matrices

The unreduced number k is the smallest integer such that the next superdiagonal element is includable in the semiseparable structure, or such that the lower left block is zero.

For this example the unreduced number is 2.



The implicit Q-theorem for semiseparable matrices

Then we have that for:

$$\begin{cases} Q_1^T S Q_1 &= S_1 \\ Q_2^T S Q_2 &= S_2 \end{cases}$$

with the first column of Q_1 equal to the first column of Q_2, that the matrices S_1 and S_2 have the upper left $k \times k$ block equal (except for the signs of the elements).

In fact we still need to prove that $k = n$.

The implicit Q-theorem for semiseparable matrices

For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
▶ QR maintains the tridiagonal structure	▶ QR maintains the semiseparable structure
▶ Chasing technique	▶ Chasing technique
▶ Proof of correctness	

The implicit Q-theorem for semiseparable matrices

For an implicit QR-algorithm we need

Tridiagonal	Semiseparable
▶ QR-factorization.	▶ QR-factorization.
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Outline

Introduction
- Eigenvalue problem
- Semiseparable matrices
- Representation

The reduction to semiseparable form
- The reduction algorithms
- The convergence properties

QR-algorithms for semiseparable matrices
- The implicit and the explicit approach
- The implicit QR-algorithms
- Complete algorithm via semiseparable matrices

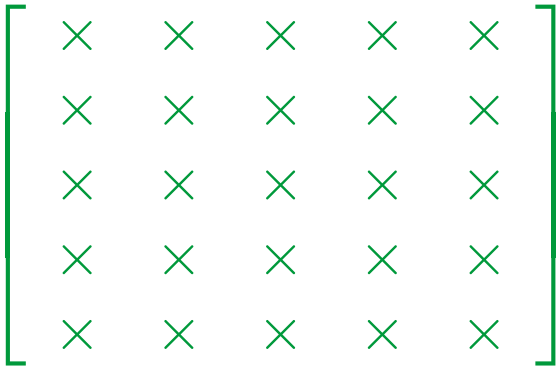
Future work

Algorithm via semiseparable matrices

Combining all the pieces we get:

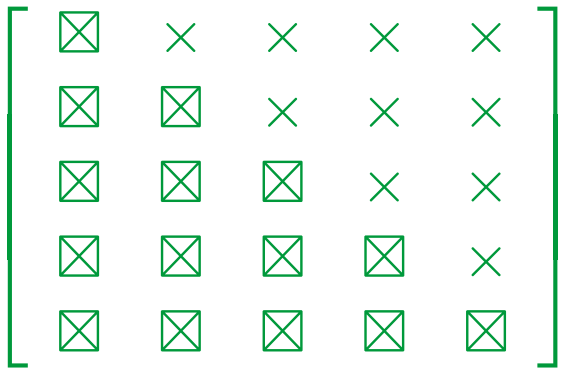
- Reduction of a matrix to semiseparable form.
- Apply the QR-algorithm until we have convergence.

Algorithm via semiseparable matrices



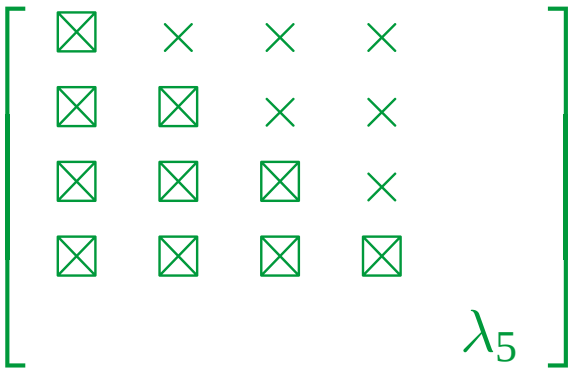
- Reduce the matrix to semiseparable form.

Algorithm via semiseparable matrices



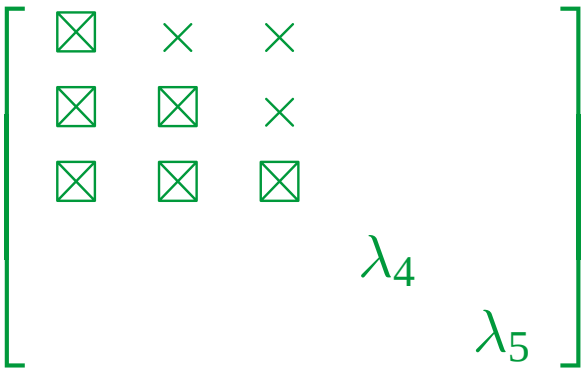
- Apply some steps of the QR-algorithm to the matrix.

Algorithm via semiseparable matrices



- Apply some steps of the QR-algorithm to the matrix.

Algorithm via semiseparable matrices



- Apply some steps of the QR-algorithm to the matrix.

Algorithm via semiseparable matrices

$$\begin{bmatrix} \boxtimes & \times & & & \\ \boxtimes & \boxtimes & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix}$$

- Apply some steps of the QR-algorithm to the matrix.

Algorithm via semiseparable matrices

$$\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix}$$

- Apply some steps of the QR-algorithm to the matrix.

Graded matrices

Calculate the eigenvalues of the graded matrix

$$A = \begin{pmatrix} 10^{40} & 10^{29} & 10^{19} \\ 10^{29} & 10^{20} & 10^{9} \\ 10^{19} & 10^{9} & 1 \end{pmatrix}$$

The eigenvalues of A are:

$$\Lambda = [1 \cdot 10^{40}, 9.9 \cdot 10^{19}, 9.818181818181818 \cdot 10^{-1}].$$

The eigenvalues computed by the routine $eig(\cdot)$ in Matlab are:

$$[-3.85544 \cdot 10^{23}, 9.90002 \cdot 10^{-01}, 1 \cdot 10^{40}].$$

Graded matrices

Reducing the matrix to semiseparable form and then applying the QR-algorithm for semiseparable matrices gives us as eigenvalues:

$$\left[1 \cdot 10^{40}, 9.9 \cdot 10^{19}, 9.818181818181789 \cdot 10^{-1}\right].$$

The original eigenvalues:

$$\Lambda = [1 \cdot 10^{40}, 9.9 \cdot 10^{19}, 9.818181818181818 \cdot 10^{-1}].$$

Conclusions

- Theoretical results for QR-algorithms.
  All the results for semiseparable, upper triangular semiseparable and Hessenberg-like
  - Unreducedness and reduction to unreduced form
  - Maintaining the semiseparable structure
  - Implicit Q-theorem for semiseparable matrices

Conclusions

- Theoretical results for QR-algorithms.
- Implicit QR-algorithm
  - For symmetric semiseparable matrices
  - For Hessenberg-like matrices
  - Singular value decomposition via upper triangular semiseparable matrices

Conclusions

- Theoretical results for QR-algorithms.
- Implicit QR-algorithm
- Software
  - All the tools presented here are available for Matlab.

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- Introduction
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- Future work

Future work

- Nullity theorem: possibilities for singular matrices.

Future work

- Nullity theorem: possibilities for singular matrices.
- Reduction to semiseparable plus diagonal

Future work

- Nullity theorem: possibilities for singular matrices.
- Reduction to semiseparable plus diagonal
- More detailed convergence analysis

Future work

- Nullity theorem: possibilities for singular matrices.
- Reduction to semiseparable plus diagonal
- More detailed convergence analysis
- More detailed complexity analysis

Future work

- Nullity theorem: possibilities for singular matrices.
- Reduction to semiseparable plus diagonal
- More detailed convergence analysis
- More detailed complexity analysis
- More general classes of structured rank matrices
  - Representation
  - QR-algorithm